Limits and Derivatives

Case Study Based Questions

Read the following passages and answer the questions that follow:

1. A function f is said to be a rational function, if

$f(x) = \frac{g(x)}{h(x)}$, where g(x) and h(x) are polynomial functions such that $h(x) \neq 0$.

Then, $\lim_{x \to a} f(x) = \lim_{x \to a} \frac{g(x)}{h(x)}$

$$= \frac{\lim_{x \to a} g(x)}{\lim_{x \to a} h(x)} = \frac{g(a)}{h(a)}$$

However, if h(a) = 0, then there are two cases arise,

(i)g(a)#0

(ii) g(a) = 0

In the first case, we say that the limit does not exist.

In the second case, we can find limit.

(A) Find the value of $\lim_{x \to -2} \left[\frac{x^2 - 4}{x^3 - 4x^2 + 4x} \right]$.

(B) Find $\lim_{x \to -1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$.

(C) Find the positive integer "n" so that

$$\lim_{x \to 3} \left[\frac{(x^n - 3^n)}{(x - 3)} \right] = 108.$$

Ans.

(A) Consider $f(x) = \frac{x^2 - 4}{x^3 - 4x^2 + 4x}$

On putting $x = 2$, we get

$f(2) = \frac{4 - 4}{8 - 16 + 8} = \frac{0}{0}$ i.e., it is of the form $\frac{0}{0}$.

So, let us first factorise it.

$$\text{Consider, } \lim_{x \to 2} \frac{x^2-4}{x^3-4x^2+4x}$$

$$= \lim_{x \to 2} \frac{(x+2)(x-2)}{x(x-2)^2}$$

$$= \lim_{x \to 2} \frac{(x+2)}{x(x-2)}$$

$$= \frac{2+2}{2(2-2)} = \frac{4}{0}$$

which is not defined.

$$\therefore \quad \lim_{x \to 2} \left[\frac{x^2-4}{x^3-4x^2+4x} \right] \text{does not exist.}$$

(B)

$$\text{Given } \lim_{x \to 1} \frac{x^7-2x^5+1}{x^3-3x^2+2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 1} \frac{x^7-x^5-x^5+1}{x^3-x^2-2x^2+2}$$

$$= \lim_{x \to 1} \frac{x^5(x^2-1)-1(x^5-1)}{x^2(x-1)-2(x^2-1)}$$

On dividing numerator and denominator by (x-1), we get

$$= \lim_{x \to 1} \frac{\frac{x^5(x^2-1)}{(x-1)} - \frac{1(x^5-1)}{(x-1)}}{\frac{x^2(x-1)}{(x-1)} - \frac{2(x^2-1)}{(x-1)}}$$

$$= \frac{\lim_{x \to 1} x^5(x+1) - \lim_{x \to 1} \left(\frac{x^5-1}{x-1}\right)}{\lim_{x \to 1} x^2 - \lim_{x \to 1} 2(x+1)}$$

$$= \frac{1\times2-5\times(1)^4}{1-2\times2} = \frac{2-5}{1-4}$$

$$\left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$= \frac{-3}{-3} = 1$$

(C)

Given limit: $\lim_{x\to3}\left[\frac{(x^n-3^n)}{(x-3)}\right] = 108$

Now, we have

$$\lim_{x\to3}\left[\frac{(x^n-3^n)}{(x-3)}\right] = n(3)^{n-1}$$

$$n(3)^{n-1} = 108$$

Now, this can be written as:

$$n(3)^{n-1} = 4(27) = 4(3)^{4-1}$$

Therefore, by comparing the exponents in the above equation, we get:

n = 4

Therefore, the value of positive integer "n" is 4.

2. Raj was learning limit of a polynomial function from his tutor Rajesh.

His tutor told that a function f is said to be a polynomial function, if f(x) is zero function.

Now, let

$f(x) = a_0 + a_1x + a_2x^2 + + anxn$ be a polynomial

function, where a's are real numbers and $a_n \# 0$.

Then, limit of a polynomial function f(x)

$= \lim_{x \to a} f(x)$

$= \lim_{x \to a} [a_0 + a_1 x + a_2 x^2 + ... + a_n x^n]$

$= \lim_{x \to a} a_0 + \lim_{x \to a} a_1 x + \lim_{x \to a} a_2 x^2 + ... + \lim_{x \to a} a_n x^n$

$= a_0 + a_1 \lim_{x \to a} x + a_2 \lim_{x \to a} x^2 + ... + a_n \lim_{x \to a} x^n$

$= a_0 + a_1 a + a_2 a^2 + ... + a_n a^n = f(a)$

Based on above information, answer the following questions.

(A) lim $(1+x+x^2+...+x^9)$ is equal to: X→--1

(a) 0

(b) 1

(c) 2

(d) 3

(B) lim $[x^2(x-1)]$ is equal to: x→5

(a) 10

(b) 100

(c) 25

(d) 125

(C) lim $(x^3 + x^2+x-1)$ is equal to: x→2

(a) 9

(b) 11

(c) 10

(d) 13

(D) lim (x^3+x+2) is equal to: X→-3

(a) 28

(b) -28

(c) 30

(d) -15

(E) lim (x^4-x^3) is equal to: X→4

(a) 192

(b) 180

(c) 50

(d) 165

Ans. (A) (a) 0

Explanation: Given, limit

$= \lim (1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9)$

$= 1-1+1-1+1-1+1-1+1-1=0$

(B) (b) 100

Explanation: Given, limit

$$= \lim_{x\to 5} [x^2(x-1)]$$

$$= \lim_{x\to 5} [x^3 - x^2]$$

$$= \lim_{x\to 5} x^3 - \lim_{x\to 5} x^2$$

$$= (5)^3 - (5)^2$$

$$= 125 - 25 = 100$$

(C) (d) 13

Explanation: Given, limit

$$= \lim_{x\to 2} (x^3 + x^2 + x - 1)$$

$$= \lim_{x\to 2} x^3 + \lim_{x\to 2} x^2 + \lim_{x\to 2} x + \lim_{x\to 2} (-1)$$

$$= (2)^3 + (2)^2 + (2) - 1$$

$$= 8 + 4 + 2 - 1 = 13$$

(D) (b) -28

Explanation: Given, limit

$$= \lim_{x\to -3} (x^3 + x + 2)$$

$$= \lim_{x\to -3} x^3 + \lim_{x\to -3} x + \lim_{x\to -3} 2$$

$=(-3)^3 + (-3)+2$

$=-27-3+2$

$=-30+228$

(E) (a) 192

Explanation: Given, limit

$$= \lim_{x\to 4} (x^4 - x^3)$$

$$= \lim_{x\to 4} x^4 - \lim_{x\to 4} x^3 = (4)^4 - (4)^3$$

$$= 256 - 64 = 192$$

3. Let f be a real valued function, the function

defined by $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ *whenever*

the limit exists is defined to be the derivative off at x.

For a function, f(x) = tan x, answer the following questions.

(A) Find the value of f(x + h)-f(x).

(B) Find the value of $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

(C) Find the derivative of f(x) = x^3 using the first principle.

Ans. (A) f(x + h) - f(x) = tan (x + h) - tan x

$$= \frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos x . \cos(x+h)}$$

$$= \frac{\sin(x+h-x)}{\cos x . \cos(x+h)} = \frac{\sin h}{\cos x \cos(x+h)}$$

(B)

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \frac{1}{\cos x \cdot \cos(x+h)}$$

$$= \lim_{h \to 0} \frac{1}{\cos x \cdot \cos(x+h)} \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= \frac{1}{\cos^2 x}$$

(C) By definition,

$$f'(x) = \lim_{h \to 0} \frac{[f(x+h)-f(x)]}{h}$$

Now, substitute f(x) = x3 in the above equation:

$$f'(x) = \lim_{h \to 0} \frac{[(x+h)^3 - x^3]}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{(x^3 + h^3 + 3xh(x+h) - x^3)}{h}$$

$$f'(x) = \lim_{h \to 0} (h^3 + 3x(x+h))$$

Substitute $h = 0$, we get

$$f'(x) = 3x^2$$

Therefore, the derivative of the function $f'(x) = x^3$ is $3x^2$.

4. Let f be a real valued function, the function

defined by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

For a function f(x) = sin x + cos x, answer the following questions.

(A) The derivative of f(x) w.r.tx is:

(a) cos x + sin x

(b) cos x - sin x

(c) 0

(d) none of these

(B) The value of f'(0) is:

(a) 2

(b) 0

(c) 1

(d) -1

(C) The value of f'(90°) is:

(a) 2

(b) 0

(c) 1

(d) -1

(D) The value of f'(60°) is:

(a) 2

(b) 0

(c) 1

(d) -1

(E) The derivative of x^2 cos x is:

(a) $2x \sin x - x^2 \sin x$

(b) $2x \cos x - x^2 \sin x$

(c) $2x \sin x - x^2 \cos x$

(d) $\cos x - x^2 \sin x \cos x$

Ans: (A) (b) cos x - sin x

Explanation: Since f(x) = sin x + cos x

Hence derivative, f'(x) = cos x - sin x

(B) (c) 1

Explanation: f'(x) = cos x - sin x

f'(0) = cos 0°- sin 0°

= 1-0

= 1

(C) (d) -1

Explanation: f'(x)= cos x - sin x

f'(90) = cos 90° - sin 90°

=0-1=-1

(D) (d)-1

Explanation: f'(x) = cos x-sin x

f'(60) = cos 60-sin 60

$= \frac{1}{2} - \frac{3}{2}$

$= -1$

(E) (b) $2x \cos x - x^2 \sin x$

Explanation: $\frac{d}{dx}(x^2 \cos x)$

Using the formula

$$\frac{d}{dx}[f(x)g(x)]$$

$$= f(x)\left[\frac{d}{dx}g(x)\right] + g(x)\left[\frac{d}{dx}f(x)\right]$$

$$\frac{d}{dx}(x^2 \cos x)$$

$$= x^2\left[\frac{d}{dx}(\cos x)\right] + \cos x\left[\frac{d}{dx}x^2\right]$$

$$= x^2(-\sin x) + \cos x(2x)$$

$$= 2x\cos x - x^2 \sin x$$